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On C_α-compact subsets

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Abstract

For an infinite cardinal α, we say that a subset B of a space X is C_α-compact in X if for every continuous function $f: X \to \mathbb{R}^\alpha$, $f[B]$ is a compact subset of $\mathbb{R}^\alpha$. This concept slightly generalizes the notion of α-pseudocompactness introduced by J.F. Kennison: a space X is α-pseudocompact if X is C_α-compact in itself. If $\alpha = \omega$, then we say C-compact instead of C_ω-compact and ω-pseudocompactness agrees with pseudocompactness. We generalize Tamano's theorem on the pseudocompactness of a product of two spaces as follows: let $A \subseteq X$ and $B \subseteq Y$ be such that A is z-embedded in X. Then the following three conditions are equivalent: (1) $A \times B$ is C_α-compact in $X \times Y$; (2) A and B are C_α-compact in X and Y, respectively, and the projection map $\pi: X \times Y \to X$ is a z_α-map with respect to $A \times B$ and A; and (3) A and B are C_α-compact in X and Y, respectively, and the projection map $\pi: X \times Y \to X$ is a strongly z_α-map with respect to $A \times B$ and A (the z_α-maps and the strongly z_α-maps are natural generalizations of the z-maps and the strongly z-maps, respectively). The degree of C_α-compactness of a C-compact subset B of a space X is defined by: $\rho(B, X) = \infty$ if B is compact, and if B is not compact, then $\rho(B, X) = \sup\{\alpha\colon B \text{ is } C_\alpha\text{-compact in } X\}$. We estimate the degree of pseudocompactness of locally compact pseudocompact spaces, topological products and $\sum$-products. We also establish the relation between the pseudocompact degree and some other cardinal functions. In the context of uniform spaces, we show that if A is a bounded subset of a uniform space $(X, \mathcal{U})$, then A is C_α-compact in $\widehat{X}$, where $(\widehat{X}, \widehat{\mathcal{U}})$ is the completion of $(X, \mathcal{U})$ iff $f(A)$ is a compact subset of $\mathbb{R}^\alpha$ from every uniformly continuous function from X into $\mathbb{R}^\alpha$; we characterize the C_α-compact subsets of topological groups; and we also prove that if $\{G_i\colon i \in I\}$ is a set of topological groups and A_i is a C_α-compact subset of G_α for all $i \in I$, then $\prod_{i\in I} A_i$ is a C_α-compact subset of $\prod_{i\in I} G_i$. © 1997 Elsevier Science B.V.

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0. Introduction

All spaces are assumed to be Tychonoff. For a space X, $\mathcal{Z}(X)$ will denote the set of zero-sets of X and for $x \in X$, $\mathcal{N}(x)$ is the set of all neighborhoods of x in X. If $f: X \to Y$ is a continuous function, then $\beta(f): \beta(X) \to \beta(Y)$ denotes the Stone–Čech extension of f. The Greek letters α, γ and κ will stand for infinite cardinal numbers. If X is a set and α is a cardinal, then $[X]^{\leqslant\alpha} = \{A \subseteq X\colon |A| \leqslant \alpha\}$. If α is a cardinal number, then α also stands for the discrete space of cardinality α. We know that $\beta(\alpha)$ can be identified with the set of all ultrafilters on α and its remainder $\alpha^* = \beta(\alpha) \setminus \alpha$ with the set of all free ultrafilters on α. If $\gamma \leqslant \alpha$, then $\mathcal{F}_\gamma(\alpha)$ will stand for the filter $\{A \subseteq \alpha\colon |\alpha \setminus A| < \gamma\}$. Observe that if $\kappa \leqslant \gamma \leqslant \alpha$, then $\mathcal{F}_\kappa(\alpha) \subseteq \mathcal{F}_\gamma(\alpha)$. For an ordinal number θ, $[0, \theta)$ will denote the space that consists of the underlying set $\{\mu\colon \mu < \theta\}$ equipped with the order topology. The space $[0, \theta + 1)$ will be denoted by $[0, \theta]$. The weight and the density of a space X are denoted by $w(X)$ and $d(X)$, respectively. A space X is said to be α-Lindelöf if every open cover of X contains a subcover of cardinality not bigger than α. The Lindelöf number $\ell(X)$ of a space X is the smallest cardinal α such that X is α-Lindelöf. For a space X, the set of all real-valued continuous functions defined on X will be denoted by $C(X)$ and, if α is a cardinal, then $C(X, \mathbb{R}^\alpha)$ will denote the set of all continuous functions from X to $\mathbb{R}^\alpha$. For a cardinal α, a subset G of a space X is called a G_α-set if G is the intersection of α-many open subsets of X (the G_ω-sets are usually called G_δ-sets). If A is a subset of a space X and α is a cardinal, then the *G_α-closure* of A in X is defined by

$$G_\alpha\text{-}\mathrm{cl}_X(A) = \{x \in X\colon \text{ if } G \text{ is a } G_\alpha\text{-set of } X \text{ and } x \in G, \text{ then } G \cap A \neq \emptyset\}.$$

We simply write G_α-cl(A) if ambiguity is impossible. We say that D is G_α-dense in X if $G_\alpha\text{-}\mathrm{cl}_X(D) = X$. Notice that if $\alpha < \gamma$, then $G_\gamma\text{-}\mathrm{cl}_X(A) \subseteq G_\alpha\text{-}\mathrm{cl}_X(A)$ for a subset A of a space X. If $r \in \mathbb{R}$, then r^* is the point of $\mathbb{R}^\alpha$ with all coordinates equal to r. If $f: X \to \mathbb{R}^\alpha$ is a continuous function, then for every $\xi < \alpha$ we write $f_\xi = \pi_\xi \circ f$, where $\pi_\xi: \mathbb{R}^\alpha \to \mathbb{R}$ is the projection map on the ξth coordinate.

In [23], Hewitt studied the spaces X such that $f[X]$ is a bounded subset of $\mathbb{R}$ for each $f \in C(X)$: he called these spaces *pseudocompact*. Besides, he proved that the pseudocompactness of a space X is equivalent to each one of the following statements:

(1) $f[X]$ is a compact subset of $\mathbb{R}$ for each $f \in C(X)$;

(2) X is G_δ-dense in $\beta(X)$;

(3) X is G_δ-dense in every compactification of X.

Hewitt's concept has been generalized, in different ways, by several topologists (see [2,5–7,17,24,26]); in particular, a subset B of a space X is called *bounded*, in X, provided that $f[B]$ is a bounded subset of $\mathbb{R}$ for all $f \in C(X)$: the boundedness of a subset B of a space X is equivalent to the condition that if $\mathcal{U}$ is a locally finite family of open subsets of X such that each one of them meets B, then $\mathcal{U}$ is finite. It should be remarked that a subset B of a space X is bounded if and only if for every cardinal α and for every $f \in C(X, \mathbb{R}^\alpha)$, there is a set $\{[a_\xi, b_\xi]\colon \xi < \alpha\}$ of closed intervals of $\mathbb{R}$ such that $f[B] \subseteq \prod_{\xi<\alpha}[a_\xi, b_\xi]$. In parallel, Isiwata [24] introduced and investigated the subsets B of a space X with the property that $\inf\{f(x)\colon x \in B\} > 0$ for every $f \in C(X)$ which is positive on B, equivalently, $f[B]$ is compact for every $f \in C(X)$: this concept is named hyperbounded by Buchwalter [7] and C-compact by the authors of [17]. We adopt the terminology from [17].

In Section 1, we give the basic properties of C_α-compact subsets and a generalization, in the context of C_α-compactness, of Tamano's theorem [34]: $X \times Y$ is pseudocompact iff both X and Y are pseudocompact and the projection map $\pi_X : X \times Y \to X$ is z-closed. The degree of C_α-compactness of a C-compact subset is introduced and estimated on locally compact, pseudocompact spaces in Section 2. In Section 3, we study the C_α-compact subsets of the completion of a uniform space and of topological groups. The degree of pseudocompactness of topological products and their $\sum$-products are studied in Section 4.

1. C_α-compact subsets

We start with a very natural generalization of C-compactness.

Definition 1.1. Let X be a space. A subset B of X is said to be C_α-compact in X if $f[B]$ is a compact subset of $\mathbb{R}^\alpha$ for every $f \in C(X, \mathbb{R}^\alpha)$.

It is not hard to see that a subset B of a space X is C-compact if and only if it is C_ω-compact in X, and if α and γ are cardinals with $\alpha < \gamma$, then every C_γ-compact subset is a C_α-compact subset. Thus, every C_α-compact subset is C-compact. If X is C_α-compact in itself (equivalently, in $\beta(X)$), then we say that X is *α-pseudocompact*: this concept was introduced by Kennison in [26]. So we have that every α-pseudocompact space is pseudocompact for any cardinal number α. It is known that $[0, \omega_1)$ is a pseudocompact space that is not ω_1-pseudocompact. Proposition 2.7 of [5], Lemma 2.4 of [17] and Theorem 1 of [29] have the following C_α-compact version (recall that a subset A of a space X is said to be *z-embedded* in X if every zero-set of A is the restriction of some zero-set of X).

Theorem 1.2. *For a subset B of X, the following are equivalent:*

(1) *B is C_α-compact in X;*
(2) *B is C_α-compact in $\beta(X)$;*
(3) *B is G_α-dense in* $\mathrm{cl}_{\beta(X)}(B)$;
(4) *B is G_α-dense in* $\mathrm{cl}_{\upsilon(X)}(B)$;
(5) *B is G_α-dense in* $\mathrm{cl}_{K(X)}(B)$ *for every compactification $K(X)$ of X;*
(6) *every cover of B of cardinality $\leqslant \alpha$ consisting of cozero sets of X has a finite subcover;*
(7) *if $\{Z_\xi\colon \xi < \alpha\} \subseteq \mathcal{Z}(X)$ and $B \cap \bigcap_{\xi\in I} Z_\xi \neq \emptyset$ for every finite subset I of α, then $B \cap \bigcap_{\xi<\alpha} Z_\xi \neq \emptyset$;*

(8) *if* $0^* \in \mathrm{cl}_{\mathbb{R}^\alpha}(f[B])$ *for a continuous function* $f : X \to \mathbb{R}^\alpha$, *then* $0^* \in f[B]$;
(9) $f[B]$ *is a closed subset of* $\mathbb{R}^\alpha$ *for all continuous functions* $f : X \to \mathbb{R}^\alpha$;
(10) *for every z-embedded subset* S *of* X *that contains* B, B *is* C_α*-compact in* S;
(11) *for every cozero subset* C *of* X *that contains* B, B *is* C_α*-compact in* C;
(12) *for every continuous function* $f : X \to Y$ *with* $w(Y) \leqslant \alpha$, $f[B]$ *is a compact subset of* Y.

We then have that B is C_α-compact in the space X iff $G_\alpha\text{-}\mathrm{cl}_{\beta(X)}(B) = \mathrm{cl}_{\beta(X)}(B)$. In the next corollary we state two useful conditions equivalent to α-pseudocompactness. Before we state it we recall that a family of subsets of a set is said to have the α-*intersection* property if every subfamily of cardinality not bigger than α has nonempty intersection.

Corollary 1.3. *For a space* X *the following are equivalent:*
(1) X *is* α*-pseudocompact;*
(2) *every family of zero sets having the finite intersection property has the* α*-intersection property;*
(3) *for every function* $f : X \to [0,1]^\alpha$, $\mathrm{cl}_{\beta(X)}(f^{-1}(x)) = (\beta(f))^{-1}(x)$ *for all* $x \in [0,1]^\alpha$.

The equivalence (1) $\Leftrightarrow$ (2) of the corollary just given was proved in [26]. The next result extends Corollary 2.9 of [17].

Lemma 1.4. *Let* X *be an* α*-pseudocompact space. If* $C = \bigcap_{\xi<\alpha} Z_\xi$, *where* $Z_\xi \in \mathcal{Z}(X)$ *for each* $\xi < \alpha$, *then* C *is* C_α*-compact in* X *as well.*

Now, we give a necessary condition to separate a particular subset of a space from a C_α-compact subset.

Lemma 1.5. *Let* X *be a space. If* $A = \bigcap_{\xi<\alpha} Z_\xi$, *where* $Z_\xi \in \mathcal{Z}(X)$ *for each* $\xi < \alpha$, B *is* C_α*-compact in* X *and* $A \cap B = \emptyset$, *then* A *and* B *are completely separated.*

Proof. For each $\xi < \alpha$, choose a continuous function $f_\xi : X \to \mathbb{R}$ such that $f_\xi^{-1}(0) = Z_\xi$. Let $f : X \to \mathbb{R}^\alpha$ be the evaluation map of the set $\{f_\xi\colon \xi < \alpha\}$. Then $f(z) = (f_\xi(z))_{\xi<\alpha}$ for $z \in X$. Then we have that $A = f^{-1}(0^*)$. Since B is C_α-compact in X and disjoint from A, $f[B]$ is a compact subset of $\mathbb{R}^\alpha$ which does not contain 0^*. Hence, we can find a continuous function $g : \mathbb{R}^\alpha \to \mathbb{R}$ such that $g(0^*) = 0$ and $g[f[B]] = \{1\}$. Then, $g \circ f$ witnesses that A and B are completely separated. $\square$

Definition 1.6. A continuous surjection $f : X \to Y$ is said to be a z_α-*map* with respect to $A \subseteq X$ and $B \subseteq Y$ if $f[A] = B$ and $f[\bigcap_{\xi<\alpha} Z_\xi \cap A]$ is a closed subset of B provided that $Z_\xi \in \mathcal{Z}(X)$ for each $\xi < \alpha$. In addition, if $f[\bigcap_{\xi<\alpha} Z_\xi \cap A] = \bigcap_{\xi<\alpha} K_\xi$, where $K_\xi \in \mathcal{Z}(B)$ for each $\xi < \alpha$, then we say that f is a *strongly* z_α*-map* with respect to A and B.

N. Noble [28] studied the projections $\pi : X \times Y \to X$ which are z-maps with respect to $A \times B$ and A, where $A \subseteq X$ and $B \subseteq Y$, and called them relatively z-closed. If $\alpha = \omega$, $X = A$ and $B = Y$, then we simply say z-map and strongly z-map: the z-maps were introduced by Frolík in [16] and the strongly z-maps were studied in [19]. Tamano's theorem (see [34], [10, Theorem 4.1]) on pseudocompactness of a product of two spaces has been generalized by Noble [28] as follows: if $A \subseteq X$ with a nonisolated point and $B \subseteq Y$, then $A \times B$ is bounded in $X \times Y$ if and only if the projection map $\pi : X \times Y \to X$ is a z-map with respect to $A \times B$ and A, A is bounded in X and B is bounded in Y. Now, we shall extend Tamano's theorem in the context of C_α-compactness. First, we prove some preliminary lemmas.

Lemma 1.7. *Let $A \subseteq X$ and $B \subseteq Y$. If $A \times B$ is C_α-compact in $X \times Y$, then*

$$\mathrm{cl}_X\left(\pi\left[\bigcap_{\xi<\alpha} Z_\xi \cap (A \times B)\right]\right) = \pi\left[\bigcap_{\xi<\alpha} Z_\xi \cap (A \times B)\right] \subseteq A,$$

where $\pi : X \times Y \to X$ is the projection map and $Z_\xi \in \mathcal{Z}(X \times Y)$ for each $\xi < \alpha$; that is, π is a z_α-map with respect to $A \times B$ and A.

Proof. Put $Z = \bigcap_{\xi<\alpha} Z_\xi$, where $Z_\xi \in \mathcal{Z}(X \times Y)$ for each $\xi < \alpha$. Suppose that

$$x \in \mathrm{cl}_X\big(\pi[Z \cap (A \times B)]\big) \setminus \pi[Z \cap (A \times B)].$$

We have that $(\{x\} \times B) \cap Z = \emptyset$ and since $\{x\} \times B$ is C_α-compact in $X \times Y$, by Lemma 1.5, there is a continuous function $f : X \times Y \to [0,1]$ such that $f((x,b)) = 1$ for all $b \in B$ and $Z \subseteq f^{-1}(0)$. Arguing as in the proof of Theorem 4.1((a) $\Leftrightarrow$ (b)) of [10], we obtain a contradiction. □

In the next three lemmas, we generalize a result that is included in the proof of 3.4(a) and 3.5 of [19]. In order to prove them we shall slightly modify the arguments given in the original proofs from [19].

Lemma 1.8. *If $f : X \to Y$ is a z-map with respect to $A \subseteq X$ and $B \subseteq Y$ such that its restriction $f : A \to B$ is an open map, $Z \in \mathcal{Z}(X)$ and C is a cozero set of X such that $Z \subseteq C$, then there is $K \in \mathcal{Z}(B)$ for which $f[Z \cap A] \subseteq K \subseteq f[C \cap A]$.*

Proof. Let $h : X \to [0,1]$ be a continuous function such that

$$Z = h^{-1}(0) \quad \text{and} \quad X \setminus C = h^{-1}(1).$$

Set $D = \{r \in (0,1)\colon r \text{ is dyadic}\}$. Now, for each $r \in D$ we put

$$U_r = f\big[h^{-1}([0,r)) \cap A\big].$$

Then, U_r is an open subset of A and $A = \bigcup_{r \in D} U_r$. Since f is a z-map with respect to A and B, $f[h^{-1}([0,r]) \cap A]$ is a closed subset of B and hence

$$\mathrm{cl}_B(U_r) \subseteq f\big[h^{-1}([0,r]) \cap A\big] \subseteq U_s$$

whenever $r, s \in D$ and $r < s$. By Lemma 3.12 of [20], $g(y) = \inf\{r \in D\colon\ y \in U_r\}$ for $y \in B$ defines a continuous function from B to $[0,1]$. If $K = g^{-1}(0)$, then $K \in \mathcal{Z}(B)$ and $f[Z \cap A] \subseteq K \subseteq f[C \cap A]$. $\square$

Lemma 1.9. *Let $f : X \to Y$ be a z-map with respect to $A \subseteq X$ and $B \subseteq Y$ such that its restriction $f : A \to B$ is an open map and $f^{-1}(b) \cap A$ is C_α-compact in X for all $b \in B$. If $h : X \to [0,1]^\alpha$ is a continuous function and $0 < n < \omega$, then there is a set $\{K_\xi\colon\ \xi < \alpha\} \subseteq \mathcal{Z}(B)$ such that*

$$f\left[h^{-1}(0^*) \cap A\right] \subseteq \bigcap_{\xi<\alpha} K_\xi \subseteq f\left[h^{-1}\left(\left[0, \frac{1}{n}\right)^\alpha\right) \cap A\right].$$

Proof. We proceed by transfinite induction on α. Lemma 1.8 is the case when $\alpha = \omega$. Assume that the conclusion of the lemma holds for all cardinals $\gamma < \alpha$ and for each $0 < m < \omega$. Let $h : X \to [0,1]^\alpha$ be a continuous function and $0 < n < \omega$. For each ordinal number $\mu < \alpha$, we set $h(\mu) = j_\mu \circ h : X \to [0,1]^\mu$, where $j_\mu : [0,1]^\alpha \to [0,1]^\mu$ is the projection map. By induction hypothesis, for each $\mu < \alpha$ there is a set

$$\left\{K_\xi^\mu\colon\ \xi < \mu\right\} \in \mathcal{Z}(B)$$

such that

$$f\left[h(\mu)^{-1}(0^*) \cap A\right] \subseteq \bigcap_{\xi<\mu} K_\xi^\mu \subseteq f\left[h(\mu)^{-1}\left(\left[0, \frac{1}{n+1}\right]^\mu\right) \cap A\right].$$

Hence,

$$\begin{aligned} f\left[h^{-1}(0^*) \cap A\right] &= f\left[\bigcap_{\mu<\alpha} h(\mu)^{-1}(0^*) \cap A\right] \subseteq \bigcap_{\mu<\alpha} f\left[h(\mu)^{-1}(0^*) \cap A\right] \\ &\subseteq \bigcap_{\mu<\alpha} \bigcap_{\xi<\mu} K_\xi^\mu \subseteq \bigcap_{\mu<\alpha} f\left[h(\mu)^{-1}\left(\left[0, \frac{1}{n+1}\right]^\mu\right) \cap A\right]. \end{aligned}$$

We claim that

$$\bigcap_{\mu<\alpha} f\left[h(\mu)^{-1}\left(\left[0, \frac{1}{n+1}\right]^\mu\right) \cap A\right] = f\left[\bigcap_{\mu<\alpha} h(\mu)^{-1}\left(\left[0, \frac{1}{n+1}\right]^\mu\right) \cap A\right].$$

In fact, let

$$y \in \bigcap_{\mu<\alpha} f\left[h(\mu)^{-1}\left(\left[0, \frac{1}{n+1}\right]^\mu\right) \cap A\right].$$

Then, we have that $f^{-1}(y) \cap A \cap h(\mu)^{-1}([0, 1/(n+1)]^\mu) \neq \emptyset$ for each $\mu < \alpha$. Since $h(\nu)^{-1}([0, 1/(n+1)]^\nu) \subseteq h(\mu)^{-1}([0, 1/(n+1)]^\mu)$ whenever $\mu < \nu < \alpha$ and $f^{-1}(y) \cap A$ is C_α-compact in X, by Theorem 1.2(7), we have that

$$f^{-1}(y) \cap A \cap \bigcap_{\mu<\alpha} h(\mu)^{-1}\left(\left[0, \frac{1}{n+1}\right]^\mu\right) \neq \emptyset$$

and hence

$$y \in f\Big[\bigcap_{\mu<\alpha} h(\mu)^{-1}\Big(\Big[0, \frac{1}{n+1}\Big]^{\mu}\Big) \cap A\Big] = f\Big[h^{-1}\Big(\Big[0, \frac{1}{n+1}\Big]^{\alpha}\Big) \cap A\Big].$$

Therefore,

$$\begin{aligned} f[h^{-1}(0^*) \cap A] &\subseteq \bigcap_{\mu<\alpha} \bigcap_{\xi<\mu} K_\xi^\mu \subseteq f\Big[\bigcap_{\mu<\alpha} h(\mu)^{-1}\Big(\Big[0, \frac{1}{n+1}\Big]^{\mu}\Big) \cap A\Big] \\ &= f\Big[h^{-1}\Big(\Big[0, \frac{1}{n+1}\Big]^{\alpha}\Big) \cap A\Big] \subseteq f\Big[h^{-1}\Big(\Big[0, \frac{1}{n}\Big)^{\alpha}\Big) \cap A\Big]. \qquad \square \end{aligned}$$

Lemma 1.10. *If $f: X \to Y$ is a z-map with respect to $A \subseteq X$ and $B \subseteq Y$ such that its restriction $f: A \to B$ is an open map and $f^{-1}(b) \cap A$ is C_α-compact in X for all $b \in B$, then f is a strongly z_α-map with respect to A and B.*

Proof. Let $\{Z_\xi\colon \xi < \alpha\} \subseteq \mathcal{Z}(X)$. Fix a continuous function $h: X \to [0,1]^\alpha$ such that $\bigcap_{\xi<\alpha} Z_\xi = h^{-1}(0^*)$. For each ordinal number $\mu < \alpha$, we put $h(\mu) = j_\mu \circ h: X \to [0,1]^\mu$, where $j_\mu: [0,1]^\alpha \to [0,1]^\mu$ is the projection map. Notice that $\bigcap_{\mu<\alpha} h(\mu)^{-1}(0^*) = h^{-1}(0^*)$. According to Lemma 1.9, for each $\mu < \alpha$ and for each $0 < n < \omega$ we may find $\{K_{n,\xi}^\mu\colon \xi < \mu\} \subseteq \mathcal{Z}(B)$ such that

$$f\big[h(\mu)^{-1}(0^*) \cap A\big] \subseteq \bigcap_{\xi<\mu} K_{n,\xi}^\mu \subseteq f\Big[h(\mu)^{-1}\Big(\Big[0, \frac{1}{n}\Big)^{\mu}\Big) \cap A\Big].$$

Hence,

$$\begin{aligned} f[h^{-1}(0^*) \cap A] = f\Big[\bigcap_{\mu<\alpha} h(\mu)^{-1}(0^*) \cap A\Big] &\subseteq \bigcap_{0<n<\omega} \bigcap_{\mu<\alpha} \bigcap_{\xi<\mu} K_{n,\xi}^\mu \\ &\subseteq \bigcap_{0<n<\omega} \bigcap_{\mu<\alpha} f\Big[h(\mu)^{-1}\Big(\Big[0, \frac{1}{n}\Big)^{\mu}\Big) \cap A\Big]. \end{aligned}$$

To show that

$$f[h^{-1}(0^*) \cap A] = \bigcap_{0<n<\omega} \bigcap_{\mu<\alpha} \bigcap_{\xi<\mu} K_{n,\xi}^\mu = \bigcap_{0<n<\omega} \bigcap_{\mu<\alpha} f\Big[h(\mu)^{-1}\Big(\Big[0, \frac{1}{n}\Big)^{\mu}\Big) \cap A\Big]$$

we argue as in the proof of Lemma 1.9. $\square$

A C_α-compactness version of Tamano's theorem is the following.

Theorem 1.11. *Let $A \subseteq X$ and $B \subseteq Y$. If A is a z-embedded in X, then the following are equivalent:*

(1) *$A \times B$ is C_α-compact in $X \times Y$;*

(2) *A and B are C_α-compact in X and Y, respectively, and the projection map $\pi: X \times Y \to X$ is a z_α-map with respect to $A \times B$ and A;*

(3) *A and B are C_α-compact in X and Y, respectively, and the projection map $\pi: X \times Y \to X$ is a strongly z_α-map with respect to $A \times B$ and A.*

Proof. (1) $\Rightarrow$ (2). This is Lemma 1.7.

(2) $\Rightarrow$ (3) This follows from Lemma 1.10.

(3) $\Rightarrow$ (1) We shall verify clause (7) of Theorem 1.2. Let $\{Z_\xi\colon \xi < \alpha\} \subseteq \mathcal{Z}(X)$ such that $(A \times B) \cap \bigcap_{\xi \in I} Z_\xi \neq \emptyset$ for each finite subset I of α. By assumption, $\pi[\bigcap_{\xi<\alpha} Z_\xi \cap (A \times B)] = \bigcap_{\xi<\alpha} R_\xi$, where $R_\xi \in \mathcal{Z}(A)$ for each $\xi < \alpha$. Now, for each $\xi < \alpha$, we can find $L_\xi \in \mathcal{Z}(X)$ for which $A \cap L_\xi = R_\xi$. It is then evident that every finite subfamily of $\{L_\xi\colon \xi < \alpha\}$ meets A and since A is C_α-compact in X,

$$A \cap \bigcap_{\xi} L_\xi = \bigcap_{\xi<\alpha} R_\xi = \pi\left[\bigcap_{\xi<\alpha} Z_\xi \cap (A \times B)\right] \neq \emptyset.$$

Hence, $A \times B \cap \bigcap_{\xi<\alpha} Z_\xi \neq \emptyset$. $\square$

It is pointed out in [5, Corollary 2.8] that a z-embedded subset B of a space X is C-compact in X iff B is pseudocompact. It follows from necessity in Noble's theorem quoted above and Theorem 1.11 that:

Corollary 1.12. *Let $A \subseteq X$ and $B \subseteq Y$. If A is z-embedded in X, then the following are equivalent:*

(1) *$A \times B$ is C-compact in $X \times Y$;*

(2) *$A \times B$ is bounded in $X \times Y$ and A and B are C-compact in X and Y, respectively.*

It then follows from Corollary 1.12 that if X is a pseudocompact space, then $X \times B$ is bounded in $X \times Y$ for every C-compact subset B of a space Y if and only if $X \times B$ is C-compact in $X \times Y$ for every C-compact subset B of Y. This improves Corollary 5.6 of [18].

Question 1.13. *If A and B are C_α-compact in X and Y, respectively, and the projection map $\pi : X \times Y \to X$ is a strongly z_α-map with respect to $A \times B$ and A, must $A \times B$ be C_α-compact in $X \times Y$?*

2. The degree of pseudocompactness

We know that any space X can be embedded in $\mathbb{R}^{w(X)}$. Hence, a subset B of a space X is compact if and only if B is $C_{w(X)}$-compact. This suggests that one should study the following cardinal function that, in particular, estimates the degree of pseudocompactness of a pseudocompact space.

Definition 2.1. Let X be a space. If B is a C-compact subset of X, then we define:

$$\rho(B, X) = \begin{cases} \infty & \text{if } B \text{ is compact,} \\ \sup\{\alpha\colon B \text{ is } C_\alpha\text{-compact in } X\} & \text{if } B \text{ is not compact.} \end{cases}$$

If X is pseudocompact, then we simply write $\rho(X)$ instead of $\rho(X, X)$. It should be noticed that if B is C-compact in X, then $\rho(B, X) \geqslant \omega$. Since every space X can be embedded in $\mathbb{R}^{w(X)}$, we must have that $\rho(X) \leqslant \ell(X) \leqslant w(X)$. Hence, if X is pseudocompact and noncompact, then X cannot be $\ell(X)$-pseudocompact. If X is a noncompact, pseudocompact space and $\rho(X)$ is a successor cardinal, then X is $\rho(X)$-pseudocompact. Thus, a space X is compact iff X is $\ell(X)$-pseudocompact$X$ iff X is $w(X)$-pseudocompact. By Theorem 1.2, we have that $\rho(X) = \rho(X, \beta(X))$ for every pseudocompact space X. It follows from the definition that if X is a pseudocompact space, then $\rho(X) \leqslant \rho(X, B(X))$ for any compactification $B(X)$ of X. By Theorem 1.2, we have that $\rho(X) = \sup\{\alpha : G_\alpha\text{-cl}_{\beta(X)}(X) = \beta(X)\}$ for every noncompact, pseudocompact space X.

A space X is called *initially α-compact*, for a cardinal α, if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leqslant \alpha$ has a finite subcover (this class of spaces was introduced by Y.M. Smirnov [32]). We have that a space X is initially α-compact iff every subset A of X of cardinality not bigger than α has a complete accumulation point in X (a proof of this fact is available in [33]). We omit the proof of the following easy result.

Theorem 2.2. *If X is initially α-compact, then $\alpha \leqslant \rho(X)$.*

We notice that $\rho([0, \omega_1)) = \omega$ and $[0, \omega_1)$ is initially ω-compact and if $p \in \omega^*$ satisfies that $\chi(p, \beta(\omega)) = 2^\omega$ (these ultrafilters exist in ZFC, see [8, Theorem 2.7]), then $\beta(\omega) \setminus \{p\}$ is γ-pseudocompact for every $\gamma < 2^\omega$ and it is not initially 2^ω-compact.

If X is a locally compact space, we denote by $A(X)$ the one-point compactification of X by the point ∞ (in the sense of Alexandroff). Then, for every uncountable cardinal number α we have that α is C-compact in $A(\alpha)$, α is not C_α-compact in $A(\alpha)$, $\rho(\alpha^+, A(\alpha^+)) = \alpha$ and if α is a limit cardinal, then $\rho(\alpha, A(\alpha)) = \alpha$. It is pointed out in Corollaries 3.8 and 3.9 of [17] that every locally compact, Lindelöf space X that is C-compact in $A(X)$ must be compact, and every locally compact, non-Lindelöf space X is C-compact in $A(X)$. Hence, $\rho(X, A(X)) \leqslant \ell(X)$ for every locally compact, non-Lindelöf space X. For locally compact spaces, we have the next results.

Lemma 2.3. *Let X be a locally compact, non-Lindelöf space. Then $\rho(X, A(X))$ exists, X is C_γ-compact in $A(X)$ for every $\gamma < \ell(X)$ and X cannot be $C_{\ell(X)}$-compact in $A(X)$.*

Proof. Since X is not Lindelöf, we have that $\ell(X) > \omega$. Let $\gamma < \ell(X)$. If X is not C_γ-compact in $A(X)$, then $\{\infty\} = \bigcap_{\xi<\gamma} V_\xi$, where V_ξ is an open subset of $A(X)$ for each $\xi < \gamma$, by Theorem 1.2. Hence, $X = \bigcup_{\xi<\gamma}(A(X) \setminus V_\xi)$ and this implies that $\ell(X) \leqslant \gamma$, but this is a contradiction. It is evident that X cannot be $C_{\ell(X)}$-compact in $A(X)$. $\square$

Corollary 2.4. *Let X be a locally compact space and let α be a cardinal. The following are equivalent:*

(1) $\ell(X) \leqslant \alpha$;

(2) *either X is compact or X is not C_α-compact in $A(X)$.*

Corollary 2.5. *Let X be a locally compact, non-Lindelöf space and let $\rho(X, A(X)) = \alpha$. The following assertions hold:*

(1) $\ell(X) = \alpha^+$ *if and only if* X *is* C_α*-compact in* $A(X)$.
(2) $\ell(X) = \alpha$ *if and only if* X *is not* C_α*-compact in* $A(X)$.

We remind the reader that a space X is said to be *almost-compact* if $|\beta(X)\setminus X| \leqslant 1$. We have that if X is an almost-compact space, then X is locally compact, all its powers are pseudocompact and $\rho(X) = \rho(X, A(X))$. We shall give an example of a locally compact space Z such that $\rho(Z) = \rho(Z, A(Z))$ and Z is not almost-compact (see remark after Corollary 2.12). It is well known that if α is a cardinal and $\mathrm{cf}(\alpha) > \omega$, then $[0,\alpha)$ is a countably compact space and $\beta([0,\alpha)) = [0,\alpha]$; that is, $[0,\alpha)$ is almost-compact. We also have that $\ell([0,\alpha)) = \mathrm{cf}(\alpha)$ for every cardinal α. As a consequence of this fact and Lemma 2.3, we have:

Theorem 2.6. *Let* α *be a cardinal with* $\mathrm{cf}(\alpha) > \omega$. *Then* $[0,\alpha)$ *is* γ*-pseudocompact for all* $\omega \leqslant \gamma < \mathrm{cf}(\alpha)$ *and it is not* $\mathrm{cf}(\alpha)$*-pseudocompact.*

Corollary 2.7. *For every cardinal* α, *we have that* $\rho([0,\alpha^+)) = \alpha$.

Proof. By Theorem 2.6, we have that $[0,\alpha^+)$ is α-pseudocompact and since $w([0,\alpha^+)) = \alpha^+$, we must have that $\rho([0,\alpha^+)) = \alpha$. □

Corollary 2.8. *If* γ *and* α *are cardinals with* $\gamma < \alpha$, *then* $[0,\gamma^+)$ *is* γ*-pseudocompact and is not* α*-pseudocompact.*

An example which separates the class of C_α-compact subsets for different cardinal numbers α in the context of topological groups can be found as follows: let X be a space. We denote by $F(X)$ the free topological group generated by X (see [9, 2.3 and 9.20]). It is known that X is a closed C-embedded subspace of $F(X)$. Let α and γ be cardinal numbers with $\alpha < \gamma$. Let X be an α-pseudocompact space which is not γ-pseudocompact. Since X is a closed C-embedded subset of $F(X)$, X is a closed C_α-compact subset of $F(X)$ which is not C_γ-compact.

Corollary 2.9. *For every limit cardinal number* α *with* $\mathrm{cf}(\alpha) > \omega$, *we have that*

$$\rho([0,\alpha)) = \mathrm{cf}(\alpha).$$

Proof. We know that $\beta([0,\alpha)) = [0,\alpha]$. According to Theorem 2.6, $[0,\alpha)$ cannot be $C_{\mathrm{cf}(\alpha)}$-compact in $[0,\alpha]$ and hence $\rho([0,\alpha]) \leqslant \mathrm{cf}(\alpha)$. It then follows from Theorem 2.6 that $\rho([0,\alpha)) = \mathrm{cf}(\alpha)$. □

The proof of the next corollary is left to the reader.

Corollary 2.10. *Let* α *be a cardinal with* $\mathrm{cf}(\alpha) > \omega$. *Then, we have:*
(1) $\alpha = \gamma^+$ *for some cardinal* γ *if and only if* $[0,\alpha)$ *is* $\rho([0,\alpha))$*-pseudocompact;*
(2) α *is weakly-inaccessible if and only if* $\rho([0,\alpha)) = \alpha$.

We turn now to consider the next cardinal function which is somehow related to that of Definition 2.1 and is useful in the study of the cardinal function ρ in the class of locally compact spaces.

Definition 2.11. Let X be a space. If B is a subset of X, then we define:

$$\rho^*(B,X) = \begin{cases} \infty & \text{if } B \text{ is compact,} \\ \min\{\alpha\colon B \text{ is not } C_\alpha\text{-compact in } X\} & \text{if } B \text{ is not compact.} \end{cases}$$

Observe that a subset A of a space X is not C-compact in X if and only if $\rho^*(A,X) = \omega$, and if B is C-compact in X, then $\rho(B,X) \leqslant \rho^*(B,X) \leqslant \rho(B,X)^+$. If X is a space, then we write $\rho^*(X) = \rho^*(X,X) = \rho^*(X,\beta(X))$. If $f : X \to Y$ is a continuous surjection, then $\rho^*(A,X) \leqslant \rho^*(f(A),Y)$ for every $A \subseteq X$ such that $f(A)$ is not compact. We also have that if X is any noncompact space, then $\rho^*(X)$ exists and $\rho^*(X) \leqslant \rho^*(X,B(X))$ for every compactification $B(X)$ of X.

Corollary 2.12. *If X is a locally compact, noncompact space, then*

$$\rho^*\big(X,A(X)\big) = \ell(X).$$

Proof. If X is Lindelöf then, by Corollary 3.8 of [17], we have that X is not C-compact in $A(X)$ and so $\rho^*(X) = \omega = \ell(X)$. Now assume that X is not Lindelöf. Since X is G_γ-dense in $A(X)$ for all $\gamma < \ell(X)$, $\ell(X) \leqslant \rho^*(X,A(X))$ and, by Lemma 2.3, $\ell(X) = \rho^*(X,A(X))$. □

For $1 \leqslant n < m < \omega$, we have that $Z = [0,\omega_n) \oplus [0,\omega_m)$ is a locally compact, noncompact space that satisfies $\rho(Z) = \omega_{n-1}$, $\rho^*(Z) = \omega_n$ and $\ell(Z) = \omega_m$. If $Z = [0,\omega_n) \oplus [0,\omega_n)$, for $n \in \omega$, then Z is a locally compact, non-almost-compact space with $\rho(Z) = \rho(Z,A(Z))$.

A space X that is C-compact in some of its compactifications is called *weakly-pseudocompact*; several properties of these spaces are investigated in [17]. It is pointed out in Corollary 2.9 of [17] that a locally compact space X is weakly-pseudocompact if and only if X is either compact or it is not Lindelöf.

Theorem 2.13. *Let X be a noncompact space. Then*

(1) *$\rho^*(X,B(X)) \leqslant \ell(X)$ for every compactification $B(X)$ of X;*

(2) *If X is a weakly-pseudocompact space, then there is a compactification $B(X)$ of X for which*

$$\rho\big(X,B(X)\big) \leqslant \rho^*\big(X,B(X)\big) \leqslant \ell(X).$$

Proof. (1) Let $B(X)$ be a compactification of X. Fix $x \in B(X) \setminus X$, then we may find a $G_{\ell(X)}$-subset H such that $x \in H$ and $H \cap X = \emptyset$. Hence, X cannot be $G_{\ell(X)}$-dense in $B(X)$ and so $\rho^*(X,B(X)) \leqslant \ell(X)$.

(2) Let $B(X)$ be a compactification of X witnessing that X is weakly-pseudocompact. By clause (1), we have that $\rho(X,B(X)) \leqslant \rho^*(X,B(X)) \leqslant \ell(X)$. □

Observe that $\rho^*(\alpha) = \omega$ and $\ell(\alpha) = \alpha$ for all cardinal numbers α.

Theorem 2.14. *For a locally compact, noncompact space X, the following are equivalent:*
(1) $\rho^*(X) = \ell(X)$;
(2) $\rho^*(X, B(X)) = \ell(X)$ *for all compactification $B(X)$ of X.*

Proof. (1) $\Rightarrow$ (2). Let $B(X)$ be a compactification of X. Then we have that

$$\ell(X) = \rho^*(X) \leqslant \rho^*\bigl(X, B(X)\bigr) \leqslant \rho^*\bigl(X, A(X)\bigr) = \ell(X),$$

the last equality follows from Corollary 2.12.
(2) $\Rightarrow$ (1). This is evident. □

Every almost-compact space satisfies the conclusion of Theorem 2.14. The space of the real numbers $\mathbb{R}$ is an example of a locally compact, non-almost-compact space with $\rho^*(\mathbb{R}) = \ell(\mathbb{R}) = \omega$.

A better upper bound for the degree of pseudocompactness is the realcompactness number of a space X which was introduced in [1] and it can be defined as follows: the realcompactness number of a space X is

$$q(X) = \min\bigl\{\alpha \geqslant \omega \colon\ X = G_\alpha\text{-cl}_{\beta(X)}(X)\bigr\}.$$

We should remark that a space X is compact iff X is $q(X)$-pseudocompact iff X is α-pseudocompact and $q(X) \geqslant \alpha$.

Theorem 2.15. *For every noncompact space X, we have*

$$\rho^*(X) \leqslant q(X) \leqslant \ell(X).$$

Hence, if X is a noncompact, pseudocompact space, then $\rho(X) \leqslant q(X)$. If X is almost-compact and noncompact, then $\rho^*(X) = q(X)$. If X is not realcompact and not pseudocompact, then $\rho^*(X) = \omega < q(X)$. For an arbitrary cardinal α, if X is pseudocompact and not α-pseudocompact, and Y is α-pseudocompact and noncompact, then $Z = X \oplus Y$ is a pseudocompact space such that $\rho^*(Z) \leqslant \alpha < q(Z)$.

The Isbell–Mrówka spaces are defined by means of an almost disjoint (AD) family of infinite subsets of ω as follows: if $\mathcal{A}$ is an AD family, then the Isbell–Mrówka space $\Psi(\mathcal{A})$ consists of the underlying set $\mathcal{A} \cup \omega$, the points of ω are isolated while if $x \in \mathcal{A}$, a basic neighborhood of x has the form $\{x\} \cup A$ where A is a cofinite subset of x. It is shown in [20, 5.I(5)], that $\Psi(\mathcal{A})$ is pseudocompact iff $\mathcal{A}$ is a maximal almost disjoint (MAD) family. It is easy to prove that if $\mathcal{A}$ is a MAD family, then $\ell(\Psi(\mathcal{A})) = |\mathcal{A}|$ and hence $\rho(\Psi(\mathcal{A})) \leqslant \rho^*(\Psi(\mathcal{A})) \leqslant |\mathcal{A}|$. Hence, $\Psi(\mathcal{A})$ is never $|\mathcal{A}|$-pseudocompact and if we assume CH, then $\rho(\Psi(\mathcal{A})) = \omega$ for every MAD family $\mathcal{A}$. We know that there are MAD families $\mathcal{A}$ for which $\Psi(\mathcal{A})$ has only one compactification (a construction of this kind of MAD families is available in [4,27,35]). If $\beta(\Psi(\mathcal{A})) = A(\Psi(\mathcal{A})$ then, by Corollary 2.12, $\rho^*(\Psi(\mathcal{A})) = \ell(\Psi(\mathcal{A})) = |\mathcal{A}|$ and hence $\Psi(\mathcal{A})$ is α-pseudocompact for every $\omega \leqslant \alpha < |\mathcal{A}|$.

3. Uniform spaces and topological groups

In this section, we will study the behavior of the product of C_α-compact subsets of topological groups. Our principal tool is the theory of uniform spaces. All the uniformities considered in this section are compatible. We shall denote the completion of a uniform space $(X,\mathcal{U})$ by $(\widehat{X},\widehat{\mathcal{U}})$ (see [15, Theorem 8.3.12]). It is known that if a space X is topologically complete, then $\mathrm{cl}_X(A)$ is compact for every bounded subset A of X.

Theorem 3.1. *Let A be a bounded subset of a uniform space $(X,\mathcal{U})$ and let $(\widehat{X},\widehat{\mathcal{U}})$ be the completion of $(X,\mathcal{U})$. The following assertions are equivalent:*

(1) *A is C_α-compact in $\widehat{X}$.*
(2) *A is C_α-compact in $\mathrm{cl}_{\widehat{X}}(A)$.*
(3) *A is G_α-dense in $\mathrm{cl}_{\widehat{X}}(A)$.*
(4) *$f[A]$ is a compact subset of $\mathbb{R}^\alpha$ for every uniformly continuous function from X into $\mathbb{R}^\alpha$.*

Proof. (1) $\Rightarrow$ (2). Let $f:\mathrm{cl}_{\widehat{X}}(A) \to \mathbb{R}^\alpha$ be a continuous function such that $f[A]$ is not compact. Since $\mathrm{cl}_{\widehat{X}}\, A$ is compact in $(\widehat{X},\widehat{\mathcal{U}})$, there exists a continuous function $g:\widehat{X} \to \mathbb{R}^\alpha$ such that $g|_A = f|_A$ and so A is not C_α-compact in $\widehat{X}$.

(2) $\Rightarrow$ (3). Suppose that there exists a set $\{Z_\xi\colon\ \xi < \alpha\}$ of zero sets in $\mathrm{cl}_{\widehat{X}}(A)$ such that $\bigcap_{\xi<\alpha} Z_\xi \cap A = \emptyset$ with $\bigcap_{\xi<\alpha} Z_\xi \neq \emptyset$. Let f_ξ be a continuous function from $\mathrm{cl}_{\widehat{X}}(A)$ into $\mathbb{R}$ such that $Z_\xi = f_\xi^{-1}(0)$, for each $\xi < \alpha$. Consider the function $f:\mathrm{cl}_{\widehat{X}}(A) \to \mathbb{R}^\alpha$ defined by $f(x) = \big(f_\xi(x)\big)_{\xi<\alpha}$ for every $x \in \mathrm{cl}_{\widehat{X}}(A)$. Pick $y \in \bigcap_{\xi<\alpha} Z_\xi$. Then

$$f(y) \in f\big[\mathrm{cl}_{\widehat{X}}(A)\big] \setminus f[A].$$

Thus, $f[A]$ is not compact.

(3) $\Rightarrow$ (4). Let f be a uniformly continuous function from X into $\mathbb{R}^\alpha$. Let $\widehat{f}$ be the uniform continuous extension of f to $\widehat{X}$. Pick $y \in \widehat{f}[\mathrm{cl}_{\widehat{X}}(A)] \setminus f[A]$. Since every point in $\mathbb{R}^\alpha$ is a G_α-point, $f^{-1}(y) \cap \mathrm{cl}_{\widehat{X}}(A)$ is a G_α-subset in $\mathrm{cl}_{\widehat{X}}(A)$ which does not meet A.

(4) $\Rightarrow$ (1). Let f be a continuous function from $\widehat{X}$ into $\mathbb{R}^\alpha$. We have that the function $f|_{\mathrm{cl}_{\widehat{X}}(A)}$ is uniformly continuous with respect to the uniform structure induced by $\widehat{\mathcal{U}}$ on $\mathrm{cl}_{\widehat{X}}(A)$, because $\mathrm{cl}_{\widehat{X}}(A)$ is compact. By Katětov's theorem [25] (also see [15, 8.5.6(b)]) there exists a uniformly continuous function g from $\widehat{X}$ into $\mathbb{R}^\alpha$ such that $g|_{\mathrm{cl}_{\widehat{X}}(A)} = f|_{\mathrm{cl}_{\widehat{X}}(A)}$. Since $g|_X$ is uniformly continuous, $g[A] = f[A]$ is compact. $\square$

The above theorem shows that it is not necessary to consider all continuous real-valued functions to study some properties of C_α-compact subsets in uniform complete spaces. Actually, it suffices to consider functions that are uniformly continuous. On the other hand, let α be an uncountable cardinal and

$$\mathcal{U} = \{\Delta\colon\ \Delta \text{ is a finite cover of } \alpha,\ \text{every element of } \Delta \text{ is either finite or cofinite}\}.$$

Then, α is not C-compact in the uniform space $(\alpha,\mathcal{U})$ and every uniformly continuous $f:(\alpha,\mathcal{U}) \to \mathbb{R}$ sends α onto a compact subset of $\mathbb{R}$. The referee communicated to the authors the following example due to A.W. Hager.

Example 3.2. An example of a bounded, non-C-compact subset B of a uniform space $(X,\mathcal{U})$ such that B is a C-compact subset of $(\widehat{X},\widehat{\mathcal{U}})$ is the following:

Let $X = [0,\omega_1) \times \beta(\omega_1) \cup \{\omega_1\} \times \omega_1$, and let $B = \{\omega_1\} \times \omega_1 \subseteq X$. Since X is pseudocompact, all of its subsets (in particular, B) are bounded in X. To see that B is not C-compact in $\beta(X) = [0,\omega_1] \times \beta(\omega_1)$ (hence, B is not C-compact in X) let f be a continuous function on $\beta(\omega_1)$ with $f[\omega_1]$ not compact, and define $F:\beta(X) \to \mathbb{R}$ by $F(x,y) = f(x)$ for all $(x,y) \in \beta(X)$; then $F|_X$ is continuous on X, and $F[B] = F[\omega_1]$ is not compact. We note that X is locally compact. Let $\mathcal{U}$ be the uniformity on X induced from its one-point compactification. Then, we have that every real-valued function f uniformly continuous on X with respect to $\mathcal{U}$ has a continuous extension $\widehat{f}$ to the one-point compactification $A(X) = X \cup \{\infty_X\}$. The one-point compactification $A(B)$ of B is $B \cup \{\infty_B\} = B \cup \{\infty_X\}$, so $f[B] = \widehat{f}[B]$ is compact since B is C-compact in $A(B)$.

Lemma 1.3 of [11] can be generalized as follows:

Lemma 3.3. *Let α be a cardinal and let A_i be a subset of a nonempty space X_i for $i \in I$. Then A_i is G_α-dense in X_i, for all $i \in I$ if and only if $\prod_{i\in I} A_i$ is G_α-dense in $\prod_{i\in I} X_i$.*

The following two corollaries are immediate consequences of Theorem 3.1 and Lemma 3.3.

Corollary 3.4. *Let $\{(X_i,\mathcal{U}_i): i \in I\}$ be a set of complete uniform spaces. Let B_i be a C_α-compact subset of X_i for each $i \in I$. Then $\prod_{i\in I} B_i$ is a C_α-compact subset of $\prod_{i\in I} X_i$. In particular, the product of C_α-compact subsets of topologically complete spaces X_i is again C_α-compact in the product of the spaces X_i.*

Corollary 3.5. *Let $\{X_i: i \in I\}$ be a set of topological spaces such that*

$$\beta\Big(\prod_{i\in I} X_i\Big) = \prod_{i\in I} \beta X_i.$$

Then, $\prod_{i\in I} B_i$ is a C_α-compact subset of $\prod_{i\in I} X_i$ whenever B_i is a C_α-compact subset of X_i for every $i \in I$.

In [21], I. Glicksberg also gave the conditions for the pseudocompactness of a product in terms of countable subproducts: a product of nonempty spaces is pseudocompact iff each countable subproduct is pseudocompact. It is then natural to ask whether there is the C_α-compact version of Glicksberg's result. In the following theorem, we give a partial answer to this question. First, we state a lemma that was proved by W.W. Comfort and S. Negrepontis [12, Theorem 10.14].

Lemma 3.6. *Let α be a cardinal, let $\{X_i: i \in I\}$ be a set of spaces with each $d(X_i) \leqslant \alpha$ and let $f:\prod_{i\in I} X_i \to Y$ be a continuous function with Y a space such that $w(Y) \leqslant \alpha$.*

Then, there is a subset J of I and a continuous function $g:\prod_{i\in J} X_i \to Y$ such that $|J| \leqslant \alpha$ and $g \circ \pi_J = f$, where $\pi_J:\prod_{i\in I} X_i \to \prod_{i\in J} X_i$ is the projection map.

Theorem 3.7. *Let $\{X_i\colon\ i \in I\}$ be a set of spaces and let $B_i \subseteq X_i$ for every $i \in I$. If γ and α are cardinal numbers such that $\alpha \leqslant \gamma$ and $\sup\{d(X_i)\colon\ i \in I\} \leqslant \gamma$, then $\prod_{i\in I} B_i$ is C_α-compact in $\prod_{i\in I} X_i$ if and only if $\prod_{i\in J} B_i$ is C_α-compact in $\prod_{i\in J} X_i$ for every $J \subseteq I$ with $|J| \leqslant \gamma$.*

Proof. Let γ and α be cardinal numbers with $\alpha \leqslant \gamma$ and $\sup\{d(X_i)\colon\ i \in I\} \leqslant \gamma$. We will only prove the sufficiency. Suppose that $\prod_{i\in J} B_i$ is C_α-compact in $\prod_{i\in J} X_i$ for every $J \subseteq I$ with $|J| \leqslant \gamma$. Let $f:\prod_{i\in I} X_i \to \mathbb{R}^\alpha$ be a continuous function. Since $w(\mathbb{R}^\alpha) = \alpha \leqslant \gamma$, by Lemma 3.6, there is a subset K of I and a continuous function

$$g:\prod_{i\in K} X_i \to \mathbb{R}^\alpha$$

such that $|K| \leqslant \gamma$ and $g \circ \pi_K = f$, where

$$\pi_K:\prod_{i\in K} X_i \to \prod_{i\in K} X_i$$

is the projection map. By assumption,

$$g\left[\pi_K\left[\prod_{i\in I} B_i\right]\right] = g\left[\prod_{i\in K} B_i\right] = f\left[\prod_{i\in I} B_i\right]$$

is compact. This shows that $\prod_{i\in I} B_i$ is C_α-compact in $\prod_{i\in I} X_i$. □

We shall now study the C_α-compact subsets of topological groups. The next theorem generalizes Theorem 2 of [22]. Let G be a topological group, we denote by $\mathcal{U}_L$ (respectively $\mathcal{U}_{LR}$) the left (respectively, two-sided) uniform structure on G. Let $(\overline{G},\overline{\mathcal{U}_{LR}})$ denote the uniform completion of the uniform space $(G,\mathcal{U}_{LR})$. It is known [30, Theorem 10.12(c)] that $\overline{G}$ is a topological group.

Theorem 3.8. *Let B be a bounded subset of a topological group G. The following assertions are equivalent:*

(1) *B is C_α-compact in G;*

(2) *B is G_α-dense in $\mathrm{cl}_Y B$, with $(Y,\mathcal{V})$ a uniform space such that $G \subseteq Y$ and $\mathcal{V}|_G \geqslant \mathcal{U}_L$;*

(3) *B is G_α-dense in $\mathrm{cl}_{\overline{G}}(B)$;*

(4) *for every uniform continuous function f from $(G,\mathcal{U}_{LR})$ into $\mathbb{R}^\alpha$, $f[B]$ is compact.*

Proof. The proof of the equivalences among (1), (2) and (3) is similar to that of Theorem 2 of [22].

(3) $\Rightarrow$ (4). This implication follows from (3) $\Rightarrow$ (4) of Theorem 3.1.

(4) $\Rightarrow$ (1). Suppose there exists $f:G \to \mathbb{R}^\alpha$ such that $f[B]$ is not compact. By [36, Corollary 2.29], $f|_B$ is uniformly continuous with respect to the uniform structure

induced by $\mathcal{U}_{\mathcal{LR}}$ on B. So, we can extend f to a continuous function to $\mathrm{cl}_{\overline{G}}(B)$ which is the completion of B with respect to $\mathcal{U}_{\mathcal{LR}}$. By Katětov's theorem [25], there exists a uniformly continuous function g from $\mathcal{G}$ into $\mathbb{R}^\alpha$ such that $g|_B = f|_B$. Thus, $g[A]$ is not compact, a contradiction. □

We shall slightly generalize Corollary 3 of [22] and Theorem 1.4 of [13]. We need some notation. For a set $\{G_i\colon i \in I\}$ of topological groups, we denote by $\mathcal{U}^i_{LR}$ the two-sided uniform structure on G_i for all $i \in I$. If $G = \prod_{i\in I} G_i$ and $\mathcal{U}_{LR}$ is the two-sided uniformity on G, it is known that $\mathcal{U}_{LR} = \prod_{i\in I} \mathcal{U}^i_{LR}$ [30, Proposition 3.35].

Corollary 3.9. *Let $\{G_i\colon i \in I\}$ be a set of topological groups and let A_i be a C_α-compact subset of G_i for all $i \in I$. Then $\prod_{i\in I} A_i$ is a C_α-compact subset of $\prod_{i\in I} G_i$.*

We end this section with an example. If G is a totally bounded, nonpseudocompact topological group (for the definition of totally bounded group see [9, Section 1]), then every real-valued function that is uniformly continuous with respect to the left uniformity $\mathcal{U}_L$ on G is bounded on G (see [9, 1.13]), and there is a continuous real-valued function that is not bounded on G. The maximal totally bounded topological group topology on an infinite Abelian group is not pseudocompact (for a proof of this fact we refer the reader to [9, 9.13]).

4. Topological products and $\sum$-products

We start with an estimation of the degree of pseudocompactness of a product of a set of topological spaces.

Theorem 4.1. *Let A_i be a subset of X_i for $i \in I$. Then,*

$$\rho^*\Big(\prod_{i\in I} A_i, \prod_{i\in I} X_i\Big) \leqslant \min\big\{\rho^*(A_i, X_i)\colon\ i \in I\big\}.$$

Proof. Fix $j \in I$. Since A_j is not $C_{\rho^*(A_j,X_j)}$-compact, we have that $\prod_{i\in I} A_i$ is not $C_{\rho^*(A_j,X_j)}$-compact in $\prod_{i\in I} X_i$, and hence $\rho^*(\prod_{i\in I} A_i, \prod_{i\in I} X_i) \leqslant \rho^*(A_j, X_j)$. □

We remark that if X is a pseudocompact space with $X \times X$ is not pseudocompact, then $\rho^*(X) > \omega$ and $\rho^*(X \times X) = \omega$ (for an example of such a space see [20]).

Theorem 4.2. *Let A_i be a C-compact subset of X_i, for each $i \in I$. If $\prod_{i\in I} A_i$ is C-compact in $\prod_{i\in I} X_i$, then,*

$$\rho\Big(\prod_{i\in I} A_i, \prod_{i\in I} X_i\Big) \leqslant \min\big\{\rho(A_i, X_i)\colon\ i \in I\big\}.$$

Proof. Let $\alpha = \rho(\prod_{i\in I} A_i, \prod_{i\in I} X_i)$ and $\gamma = \min\{\rho(A_i, X_i)\colon\ i \in I\}$. Then, there is $k \in I$ such that $\gamma = \rho(A_k, X_k)$. Suppose that $\gamma < \alpha$. If there is a cardinal κ such that

$\gamma < \kappa < \alpha$, then $\prod_{i\in I} A_i$ is G_κ-dense in $\mathrm{cl}_{\prod_{i\in I} X_i}(\prod_{i\in I} A_i)$. Hence, by Lemma 3.3, A_k is G_κ-dense in $\mathrm{cl}_{\beta(X_k)}(A_k)$ and so $\kappa \leqslant \gamma$, which is a contradiction. Thus, $\alpha = \gamma^+$. Then, we have that $\prod_{i\in I} A_i$ is C_α-compact in $\prod_{i\in I} X_i$. By arguing as the previous paragraph, we obtain that A_k is C_α-compact in X_k, but this is impossible. Therefore, $\alpha \leqslant \gamma$. □

We do not know whether the equality must hold in Theorem 4.2. Next, we generalize Corollary 2.12 of [17]: the proof follows from Lemma 3.3 and Glicksberg's theorem [21] (see [37, 8.25]).

Theorem 4.3. *Let A_i be a C-compact subset of X_i, for each $i \in I$. If $\prod_{i\in I} X_i$ is pseudocompact, then,*

$$\rho\left(\prod_{i\in I} A_i, \prod_{i\in I} X_i\right) = \min\{\rho(A_i, X_i):\ i \in I\}.$$

Example 4.4. Let $X = [0,\omega_1) \times [0,\omega_1)$. Then, X is a locally compact space that is not almost-compact and $\rho(X) = \rho(X, A(X))$. If α is a regular cardinal and $\omega_1 \leqslant \alpha$, then $\rho([0,\omega_1) \times [0,\alpha)) = \omega$ and $\ell([0,\omega_1) \times [0,\alpha)) = \alpha$.

We now estimate the cardinal function ρ on some subspaces of a topological product of a set of spaces. The following theorem plays a very important role in studying the function ρ on subspaces of products (it is taken from [3]). To state the theorem we need the following terminology: let X be a space, $\mathrm{hd}(X)$ and $\chi(X)$ stand for the hereditary density and the character of X, respectively; a set $\mathcal{V}$ of open subsets of X is a *π-base* of a continuous function $f: X \to Y$ at the point $x \in X$ if for every $V \in \mathcal{N}(f(x))$, $x \in \mathrm{cl}(\bigcup\{U \in \mathcal{V}:\ f(U) \subseteq V\})$;

$$\pi\chi(f,x) = \min\{|\mathcal{V}|:\ \mathcal{V} \text{ is a } \pi\text{-base of } f \text{ at } x\};$$

and the π-character of f is $\pi\chi(f) = \sup\{\pi\chi(f,x):\ x \in X\}$. Notice that $\mathrm{hd}(X) \leqslant w(X)$ and if $f: X \to Y$ is continuous, then $\pi\chi(f) \leqslant \chi(Y)$. If $X = \prod_{i\in I} X_i$ and $K \subseteq I$, then $\pi_K: \prod_{i\in I} X_i \to \prod_{i\in K} X_i$ will denote the projection mapping.

Theorem 4.5 (Arkhangel'skii factorization theorem [3]). *Let $X = \prod_{i\in I} X_i$ and let A be a dense subset of $X = \prod_{i\in I} X_i$. If $f: A \to Y$ is a continuous function and γ is a cardinal number such that*

(1) *$\mathrm{hd}(\pi_K(A)) \leqslant \gamma$ for all $K \in [I]^{\leqslant\gamma}$;*
(2) *there is a dense subset D of A such that $\pi\chi(f,x) \leqslant \gamma$ for every $x \in D$, then there is $L \in [I]^{\leqslant\gamma}$ and a continuous function $\phi: \pi_L(A) \to Y$ for which $\phi \circ \pi_L = f$.*

Corollary 4.6. *Let $X = \prod_{i\in I} X_i$ such that $w(X_i) \leqslant \gamma \leqslant |I|$ for all $i \in I$. If A is a dense subset of X and $f: A \to Y$ is a continuous function with $\chi(Y) \leqslant \gamma$, then there are $L \subseteq I$ such that $|L| \leqslant \gamma$ and a continuous function $\phi: \pi_L(A) \to Y$ such that $\phi \circ \pi_L = f$.*

Proof. For each $K \subseteq I$ with $|K| \leqslant \gamma$, we have that $w(\prod_{i\in K} X_i) \leqslant \gamma$ and hence, $\mathrm{hd}(\pi_K(A)) \leqslant \gamma$. The conclusion now follows from the fact that $\chi(Y) \leqslant \gamma$, $\pi\chi(f) \leqslant \gamma$ and Factorization Theorem 4.5. □

From Corollary 4.6, we obtain the next generalization of Lemma 4 from [14]. For $\alpha \geqslant \omega$, we say that $Y \subseteq \prod_{i\in I} X_i$ is *α-dense* if $\pi_J(Y) = \prod_{i\in J} X_i$ for all $J \in [I]^{\leqslant\alpha}$.

Lemma 4.7. *Let α be a cardinal and let $X = \prod_{i\in I} X_i$ be a product of compact spaces of weight not bigger than α with $\alpha \leqslant |I|$. Then, for a dense subset Y of X the following are equivalent.*

(1) *Y is α-pseudocompact;*
(2) *Y is C_α-compact in X;*
(3) *Y is α-dense in X.*

Proof. (1) $\Rightarrow$ (2). This is evident.

(2) $\Rightarrow$ (3). Let $J \in [I]^{\leqslant\alpha}$. Then $\pi_J(Y)$ is dense and C_α-compact in $\prod_{i\in J} X_i$. Since $w(\prod_{i\in J} X_i) \leqslant \alpha$, by Theorem 1.2(6), $\pi_J(Y)$ must be compact and so $\pi_J(Y) = \prod_{i\in J} X_i$.

(3) $\Rightarrow$ (1). Let $f: Y \to \mathbb{R}^\alpha$ be a continuous function. By Corollary 4.6, there are $J \in [I]^{\leqslant\alpha}$ and a continuous function $\phi: \pi_J(Y) \to \mathbb{R}^\alpha$ such that $\phi\circ\pi_J = f$. The function ϕ is continuous on the compact space $\pi_J(Y) = \prod_{i\in J} X_i$, so $f(Y)$ is compact. □

By using filters, we may generalize the concept of $\sum$-product as follows:

Definition 4.8. Let α be a cardinal, $\mathcal{F}$ a filter on α, $X = \prod_{\xi<\alpha} X_\xi$ and $z \in X$. Then we define the $\sum_{\mathcal{F}}$-product of X based at z by $\sum_{\mathcal{F}}(z) = \{x \in X\colon \{\xi < \alpha\colon x_\xi = z_\xi\} \in \mathcal{F}\}$.

It is not hard to see that if $\mathcal{F}$ is a filter on α, then $\sum_{\mathcal{F}}(z) = \bigcap\{\sum_p(z)\colon p \in \beta(\alpha)$ and $\mathcal{F} \subseteq p\}$, for every $z \in \prod_{\xi<\alpha} X_\xi$. We have that if $\gamma \leqslant \alpha$, then $\sum_{\mathcal{F}_\gamma(\alpha)}(z) = \sum_\gamma(z)$, where $\sum_\gamma(z) = \{x \in X\colon |\{\xi < \alpha\colon x_\xi \neq z_\xi\}| < \gamma\}$ is the original definition of the $\sum$-product based at $z \in X = \prod_{\xi<\alpha} X_\xi$. Hence, for every filter $\mathcal{F}$ on α with $\mathcal{F}_\gamma(\alpha) \subseteq \mathcal{F}$, we obtain that $\sum_\gamma(z) \subseteq \sum_{\mathcal{F}}(z)$ for every $z \in \prod_{\xi<\alpha} X_\xi$. Notice that if $\omega \leqslant \alpha \leqslant \gamma$, then the $\sum_\gamma$-product of $\prod_{i\in I} X_i$ is α-dense.

Lemma 4.9. *Let $\mathcal{F}$ be a filter on α, let $X = \prod_{\xi<\alpha} X_\xi$ be a product of spaces having more than one point and let $z \in \prod_{\xi<\alpha} X_\xi$. Then $\sum_{\mathcal{F}}(z)$ is a dense (proper) subset of X if and only if $\mathcal{F}_\omega(\alpha) \subseteq \mathcal{F}$.*

Proof. Necessity. Assume that there is $A \in \mathcal{F}_\omega(\alpha) \setminus \mathcal{F}$. Put $\alpha \setminus A = \{\xi_i\colon i < m\}$. Then, we have that $B \setminus A \neq \emptyset$ for all $B \in \mathcal{F}$. Let $V = \bigcap_{i<m} \pi_{\xi_i}^{-1}(V_i)$, where V_i is a nonempty open subset of X_{ξ_i} such that $z_{\xi_i} \notin V_i$ for every $i < m < \omega$. By assumption, there is $x \in V \cap \sum_{\mathcal{F}}(z)$. Since $\{\xi < \alpha\colon x_\xi = z_\xi\} \in \mathcal{F}$, we can find $k < m$ so that $x_{\xi_k} = z_{\xi_k}$, but this is impossible because $x_{\xi_k} \in V_k$ and $z_{\xi_k} \notin V_k$.

Sufficiency. Let $V = \bigcap_{j<n} \pi_{\xi_j}^{-1}(V_j)$, where $\xi_j < \alpha$ and $V_j \neq \emptyset$ is an open subset of X_{ξ_j} for every $j < n < \omega$. Since $\mathcal{F}_\omega(\alpha) \subseteq \mathcal{F}$, $A = \alpha \setminus \{\xi_j\colon j < n\} \in \mathcal{F}$. It then follows that $V \cap \sum_{\mathcal{F}}(z) \neq \emptyset$. □

We turn now to the principal result concerning $\sum_{\mathcal{F}}$-products.

Theorem 4.10. *Let $X = \prod_{\xi<\alpha} X_\xi$ be a product of compact spaces having more than one point and $w(X_\xi) \leqslant \gamma \leqslant \alpha$ for all $\xi < \alpha$. Let $z \in X$ and let $\mathcal{F}$ be a filter on α such that* $\sup\{\kappa < \alpha\colon \mathcal{F}_\kappa(\alpha) \subseteq \mathcal{F}\} = \gamma$. *Then, the following are equivalent*:

(1) $\kappa < \gamma$;

(2) $\sum_{\mathcal{F}}(z)$ *is κ-pseudocompact*;

(3) $\sum_{\mathcal{F}}(z)$ *is C_κ-compact in X.*

In order to prove Theorem 4.10 we need the following lemmas.

Lemma 4.11. *Let α be an uncountable cardinal, $\omega < \gamma \leqslant \alpha$, $X = \prod_{\xi<\alpha} X_\xi$ a product of compact spaces having more than one point and $z \in X$. If $w(X_\xi) \leqslant \gamma$ for all $\xi < \alpha$, then*

(1) $\sum_\gamma(z)$ *is κ-pseudocompact for all $\omega \leqslant \kappa < \gamma$ and it is not γ-pseudocompact; and*

(2) *if $\mathcal{F}$ is a filter on α with $\mathcal{F}_\gamma(\alpha) \subseteq \mathcal{F}$, then $\sum_{\mathcal{F}}(z)$ is κ-pseudocompact for all $\omega \leqslant \kappa < \gamma$, and if $\gamma \in \mathcal{F}$, then $\gamma \leqslant \rho(\sum_{\mathcal{F}}(z))^+$.*

Proof. (1) In virtue of Corollary 10.7(b) of [12], $\beta(\sum_\gamma(z)) = X$. Let $\omega \leqslant \kappa < \gamma$. We have that the space $\sum_\gamma(z)$ is a dense subset of X and for each $K \in [I]^{\leqslant\kappa}$, $\pi_K(\sum_\gamma(z)) = \prod_{\xi\in K} X_\xi$. By Lemma 4.7, $\sum_\gamma(z)$ is κ-pseudocompact. Now, for each $\xi < \gamma$ choose $x_\xi \in X_\xi \setminus z_\xi$. Since $w(X_\xi) \leqslant \gamma$ for all $\xi < \gamma$, then $G = \bigcap_{\xi<\gamma} \pi^{-1}(x_\xi)$ is a G_γ-set in X which does not meet $\sum_\gamma(z)$. Thus, $\sum_\gamma(z)$ is not γ-pseudocompact.

(2) Since $\sum_\gamma(z) \subseteq \sum_{\mathcal{F}}(z)$, by clause (1), $\sum_{\mathcal{F}}(z)$ is κ-pseudocompact for all $\omega \leqslant \kappa < \gamma$. Assume that $\gamma \in \mathcal{F}$ and that $\rho(\sum_{\mathcal{F}}(z)) < \alpha$. Let $\gamma < \kappa < \alpha$ and let $j : \prod_{\xi<\kappa} X_\xi \to [0,1]^\kappa$ be an embedding. First, observe that $\mathcal{G} = \{A \subset \kappa\colon A \in \mathcal{F}\}$ is a filter on κ. Then $\pi_\kappa[\sum_{\mathcal{F}}(z)] = \sum_{\mathcal{G}}(y)$, where $y = \pi_\kappa(z)$. It follows from Lemma 4.9 that $\sum_{\mathcal{G}}(y)$ is not compact since $\mathcal{F}_\gamma(\kappa) \subseteq \mathcal{G}$. Hence, $j[\sum_{\mathcal{G}}(y)]$ is not compact in $[0,1]^\kappa$. So, $\sum_{\mathcal{F}}(z)$ is not κ-pseudocompact. Therefore, $\gamma \leqslant \rho(\sum_{\mathcal{F}})(z)^+$. □

Lemma 4.12. *Let $X = \prod_{\xi<\alpha} X_\xi$ be a product of compact spaces having more than one point and weight $\leqslant \gamma \leqslant \alpha$. Let $z = (z_\xi)_{\xi<\alpha} \in X$ and let p be an ultrafilter on α such that* $\sup\{\kappa < \alpha\colon \mathcal{F}_\kappa(\alpha) \subset p\} = \gamma$. *Then, the following are equivalent*:

(1) $\kappa < \gamma$;

(2) $\sum_p(z)$ *is κ-pseudocompact*;

(3) $\sum_p(z)$ *is C_κ-compact in X.*

Proof. (1) $\Rightarrow$ (2) is a consequence of Lemma 4.11 and (2) $\Rightarrow$ (3) is evident. We only need to prove (3) implies (1). By hypothesis, $\mathcal{F}_{\gamma^+}(\alpha)$ is not contained in p. Choose

$F \in \mathcal{F}_{\gamma^+}(\alpha) \setminus p$. Since $F \in \mathcal{F}_{\gamma^+}(\alpha)$ and p is an ultrafilter, $|\alpha \setminus F| \leqslant \gamma$ and $\alpha \setminus F \in p$. For each $\xi \in \alpha \setminus F$, let $x_\xi \in X_\xi \setminus \{z_\xi\}$. The set $G = \bigcap\{\pi_\xi^{-1}(x_\xi)\colon \xi \in \alpha \setminus F\}$ is a G_γ-set in X which does not intersect $\sum_p(z)$. It follows from Theorem 1.2 that $\sum_p(z)$ is not C_γ-compact in X. □

Proof of Theorem 4.10. We only have to prove (3) ⇒ (1). Let $A \in \mathcal{F}_{\gamma^+}(\alpha) \setminus \mathcal{F}$. Choose an ultrafilter p on α such that $\mathcal{F} \subset p$ and $A \in \mathcal{F}_{\gamma^+}(\alpha) \setminus p$. Now, by Lemma 4.11, we obtain that $\sum_p(z)$ is not C_γ-compact in X. Let $f\colon X \to \mathbb{R}^\gamma$ be a continuous function such that $f[\sum_p(z)]$ is not compact. Since $\sum_{\mathcal{F}}(z)$ is dense in $\sum_p(z)$, $f[\sum_{\mathcal{F}}(z)]$ is not compact. So, $\sum_{\mathcal{F}}(z)$ is not C_γ-compact in X. □

Corollary 4.13. *Let α be a cardinal. Then,*

(1) *if $\gamma < \alpha$, then $\mathbb{R}^\alpha$ contains a pseudocompact subspace Y such that $\rho(Y) = \gamma$;*
(2) *if α is a limit, then $\mathbb{R}^\alpha$ contains a pseudocompact subspace Y such that $\rho(Y) = \alpha$;*
(3) *if α is not a limit, then there is not a pseudocompact subspace Y of $\mathbb{R}^\alpha$ with $\rho(Y) = \alpha$.*

It follows from Corollary 4.13(2) that there is a noncompact, pseudocompact space X with $\rho(X) = w(X)$.

We recall for the reader that a filter $\mathcal{F}$ on a cardinal number α is said to be *γ-complete*, for $\gamma \leqslant \alpha$, if $\bigcap_{\xi<\kappa} A_\xi \in \mathcal{F}$ whenever $A_\xi \in \mathcal{F}$ for every $\xi < \kappa$ and $\kappa < \gamma$. Notice that a filter $\mathcal{F}$ is not ω_1-complete iff there is $\{A_n\colon n < \omega\} \subseteq \mathcal{F}$ such that $\bigcap_{n<\omega} A_n = \emptyset$ and $A_{n+1} \subseteq A_n$ for every $n < \omega$.

Theorem 4.14. *Let α be an uncountable cardinal and let $X = \prod_{\xi<\alpha} X_\xi$ be a product of spaces having more than one point and $z \in X$. If $\mathcal{F}$ is a filter on α which is not ω_1-complete, then $\sum_{\mathcal{F}}(z)$ cannot be countably compact.*

Proof. Fix $r \in X$ such that $r_\xi \neq z_\xi$ for all $\xi < \alpha$. For each $n < \omega$ define $y^n \in \sum_{\mathcal{F}}(z)$ by $y^n_\xi = z_\xi$ if $\xi \in A_n$ and $y^n = r_\xi$ otherwise. Suppose that $\{y^n\colon n < \omega\}$ has an accumulation point in $\sum_{\mathcal{F}}(z)$, say y. Set $A = \{\xi < \alpha\colon y_\xi = z_\xi\}$. Then, $A \in \mathcal{F}$. Pick $\zeta \in A$ and let V be an open subset of X_ζ with $z_\zeta \in V$ and $r_\zeta \notin V$. Let $m < \omega$ be such that $\zeta \notin A_m$. Then, $y^n \notin \pi_\zeta^{-1}(V)$ for every $m < n < \omega$, but this is a contradiction. □

Let $\omega < \gamma < \alpha$, let $X = \prod_{\xi<\alpha} X_\xi$ be a product of compact spaces having more than one point, $w(X_\xi) \leqslant \gamma$ for each $\xi < \alpha$ and $z \in X$. If $\mathcal{F}$ is a filter on α satisfying the conditions of Theorem 4.14 and $\mathcal{F}_\gamma(\alpha) \subseteq \mathcal{F}$, then $\sum_{\mathcal{F}}(z)$ is a γ-pseudocompact space that is not countably compact. A very interesting question that remains unsolved is the following.

Question 4.15 (T. Retta [29]). *For $\omega < \alpha$, are there α-pseudocompact spaces X and Y such that $X \times Y$ is not pseudocompact?*[3]

[3] This question has been answered in the negatively by S. García-Ferreira, M. Sanchis and S. Watson.

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References

[1] A.V. Arkhangel'skii, Functional tightness, Q-spaces and τ-embeddings, Comment. Math. Univ. Carolin. 24 (1983) 105–120.
[2] A.V. Arkhangel'skii, Function spaces in the topology of pointwise convergence, Russian Math. Surveys 39 (1984) 5–66.
[3] A.V. Arkhangel'skii, Continuous mappings, factorization theorems and function spaces, Trans. Moscow Math. Soc. 47 (1985) 1–22.
[4] A.I. Bashkirov, On Stone–Čech compactifications of Isbell spaces, Bull. Polish Acad. Sci. 27 (1979) 613–619.
[5] R.L. Blair and M.A. Swardson, Spaces with an O_Z, Stone–Čech compactification, Topology Appl. 36 (1990) 73–92.
[6] J.L. Blasco, On μ-spaces and k_r-spaces, Proc. Amer. Math. Soc. 67 (1977) 179–186.
[7] H. Buchwalter, Topologies et compactopologies, Publ. Dép. Math. 6 (1969) 1–74.
[8] W.W. Comfort, Ultrafilters: some old and some new results, Bull. Amer. Math. Soc. 83 (1977) 417–455.
[9] W.W. Comfort, Topological groups, in: K. Kunen and J. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) Chapter 24, 1143–1263.
[10] W.W. Comfort and A.W. Hager, The projection mapping and other continuous functions on a product space, Math. Scand. 28 (1971) 77–90.
[11] W.W. Comfort and S. Negrepontis, The ring $C(X)$ determines the category of X, Proc. Amer. Math. Soc. 16 (1965) 1041–1045.
[12] W.W. Comfort and S. Negrepontis, Chain Conditions in Topology, Cambridge Tracts in Math. 79 (Cambridge Univ. Press, 1982).
[13] W.W. Comfort and K.A. Ross, Pseudocompactness and uniform continuity in topological groups, Pacific J. Math. 16 (1966) 483–496.
[14] B. Efimov and R. Engelking, Remarks on dyadic spaces II, Colloq. Math. 13 (1963) 181–197.
[15] R. Engelking, General Topology, Sigma Ser. Pure Math. 6 (Heldermann, Berlin, 1989).
[16] Z. Frolík, Applications of complete family of continuous functions to the theory of Q-spaces, Czechoslovak Math. J. 11 (1961) 115–133.
[17] S. García-Ferreira and A. García-Maynez, On weakly-pseudocompact spaces, Houston J. Math. 20 (1990) 145–159.
[18] S. García-Ferreira and M. Sanchis, On C-compact subsets, submitted.
[19] A. García-Máynez, Generalizations of the concepts of closed function and C-embedded set, An. Inst. Mat. Univ. Nac. Autónoma México 24 (1984) 13–35.
[20] L. Gillman and M. Jerison, Rings of Continuous Functions, Grad. Texts in Math. 43 (Springer-Verlag, Berlin, 1976).
[21] I. Glicksberg, Stone–Čech compactifications of products, Trans. Amer. Math. Soc. 90 (1959) 369–382.
[22] S. Hernández and M. Sanchis, G_δ-open functionally bounded subsets in topological groups, Topology Appl. 53 (1993) 289–299.
[23] H. Hewitt, Rings of real-valued continuous functions I, Trans. Amer. Math. Soc. 64 (1948) 54–99.
[24] T. Isiwata, Mappings and spaces, Pacific J. Math. 20 (1967) 455–480.

[25] M. Katětov, On real-valued functions in topological spaces, Fund. Math. 38 (1951) 73–84.
[26] J.F. Kennison, m-pseudocompactness, Trans. Amer. Math. Soc. 104 (1962) 436–442.
[27] S. Mrówka, Some set-theoretic construction in topology, Fund. Math. 94 (1977) 83–92.
[28] N. Noble, Ascoli theorems and the exponential map, Trans. Amer. Math. Soc. 143 (1969) 393–411.
[29] T. Retta, Some cardinal generalizations of pseudocompactness, Czechoslovak Math. J. 43 (1993) 385–390.
[30] W. Roelcke and S. Dierolf, Uniform Structures on Topological Groups and their Quotients (McGraw-Hill, New York, 1981).
[31] V. Saks and R.M. Stephenson Jr, Products of $\mathcal{M}$-compact spaces, Proc. Amer. Math. Soc. 28 (1971) 279–288.
[32] Y.M. Smirnov, On topological spaces, compact in a given interval of powers, Izv. Akad. Nauk SSSR Ser. Mat. 14 (1950) 155–178.
[33] R.M. Stephenson Jr, Initially κ-compact and related spaces, in: K. Kunen and J. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984).
[34] H. Tamano, A note on the pseudocompactness of the product of two space, Mem. Coll. Sci. Univ. Kyoto Ser. A 33 (1960) 225–230.
[35] J. Terasawa, Spaces $\mathbb{N} \cup \mathcal{R}$ and their dimensions, Topology Appl. 11 (1980) 93–102.
[36] M.G. Tkachenko, Compactness type properties in topological groups, Czechoslovak Math. J. 38 (1988) 324–341.
[37] R.C. Walker, The Stone–Čech Compactification (Springer-Verlag, Berlin, 1974).